On an isomorphism problem for reduced finitary power monoids

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Semigroups and monoids

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- E.g. $(\mathbb{N}, \cdot)$ and $(\mathbb{N}_0, +)$ are cancellative but $(\mathbb{N}_0, \cdot)$ is not.

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- E.g. $\mathsf{q}((\mathbb{N}, \cdot)) = (\mathbb{Q}_{>0})$ and $\mathsf{q}((\mathbb{N}_0, +)) = \mathbb{Z}$.

Power semigroups

- The *large power semigroup* of a semigroup S, denoted by $\mathcal{P}(S)$, is the family of all non-empty subsets of S endowed with set multiplication

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- If H is a monoid, then $\mathcal{P}(H)$ is a monoid with identity element $\{1_H\}$ and we call $\mathcal{P}(H)$ the *large power monoid* of H.

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- Let S be a semigroup and H a monoid. We define the following subsemigroups of $\mathcal{P}(S)$ (resp. of $\mathcal{P}(H)$).
- $\mathcal{P}_{\text{fin}}(S) = \{X \in \mathcal{P}(S) : |X| < \infty\}$, the *finitary power semigroup* of S.
- $\mathcal{P}_{\times}(H) = \{X \in \mathcal{P}(H) : X \cap H^{\times} \neq \emptyset\}$, the *restricted large power monoid* of H.
- $\mathcal{P}_{1}(H) = \{X \in \mathcal{P}(H) : 1_H \in X\}$, the *reduced large power monoid* of H.
- $\mathcal{P}_{\text{fin},\times}(H) = \mathcal{P}_{\text{fin}}(H) \cap \mathcal{P}_{\times}(H)$, the *restricted finitary power monoid* of H.
- $\mathcal{P}_{\text{fin},1}(H) = \mathcal{P}_{\text{fin}}(H) \cap \mathcal{P}_{1}(H)$, the *reduced finitary power monoid* of H.

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- They play a prominent role in the study of formal languages and automata.

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Let $\mathcal{O}$ be a class of semigroups and let $S, T \in \mathcal{O}$. Prove or disprove that $\mathcal{P}(S) \simeq \mathcal{P}(T)$ if and only if $S \simeq T$?

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- The isomorphism problem was disproven for the class of all semigroups, proven for several other special classes, e.g. groups and Clifford semigroups and is still open for finite semigroups.

The isomorphism problem

Theorem (Tringali, 2025)

Let S and T be cancellative semigroups and let one of S and T be commutative. Then $\mathcal{P}(S) \simeq \mathcal{P}(T)$ if and only if $\mathcal{P}_{\text{fin}}(S) \simeq \mathcal{P}_{\text{fin}}(T)$ if and only if $S \simeq T$.

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- Any isomorphism maps cancellative elements to cancellative elements.
- Hence, every isomorphism $f : \mathcal{P}(S) \to \mathcal{P}(T)$ restricts to an isomorphism $S \to T$.
- This approach does not work in the non-commutative setting.

The Bienvenu-Geroldinger Conjecture

Conjecture (Bienvenu-Geroldinger, 2024)

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Theorem (Tringali-Yan, 202?)

Let G, H be torsion groups. Then $\mathcal{P}_{\text{fin},1}(G) \simeq \mathcal{P}_{\text{fin},1}(H)$ if and only if $G \simeq H$.

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- Tringali and Yan disproved the conjecture for arbitrary monoids but not for cancellative ones.

Some counterexamples

- The following examples show that neither commutativity nor cancellativity is transferrable via $\mathcal{P}_{\text{fin},1}$.

Example (Tringali-Yan)

Let $H = \{1_H, a\}$ with $a^2 = 1_H$ and $K = \{1_K, b\}$ with $b^2 = b$. Then

$$K \simeq \mathcal{P}_{\text{fin},1}(H) \simeq \mathcal{P}_{\text{fin},1}(K).$$

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Example

Let $H = \{1_H, a, b\}$ and $K = \{1_K, x, y\}$ with every element being idempotent, $ab = ba = a$, $xy = x$ and $yx = y$. Then $\mathcal{P}_{\mathrm{fin},1}(H) \simeq \mathcal{P}_{\mathrm{fin},1}(K)$.

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- Moreover, for every cardinal number $\kappa \geq 2$, there are non-isomorphic monoids H, K with $|H| = |K| = \kappa$ with $\mathcal{P}_{\text{fin},1}(H) \simeq \mathcal{P}_{\text{fin},1}(K)$.

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- This yields a canonical bijection $g : H \to K$, satisfying $f(\{1_H, x\}) = \{1_K, g(x)\}$ and $g(1_H) = 1_K$, called the *pullback* of f.

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- However, g is NOT necessarily a homomorphism.
- $\{1_H, a\} \cdot \{1_H, b\} = \{1_H, a, b, ab\}$ and $\{1_H, ab\}$ do not have an apparent connection.

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Lemma

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- Similarly, one can obtain

Lemma

Let H, K be cancellative monoids. Then $g(a^{-1}) = g(a)^{-1}$ for every $a \in H^\times$. In particular, we have $g(H^\times) = K^\times$.

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- If $a^n = b^m$ for some $a, b \in H$ and $n, m \in \mathbb{Z}$, not both zero, it is possible to show that $g(ab) = g(a)g(b)$.

The case $a^n = b^m$

Lemma

Let H, K be cancellative monoids, both containing at least one element of infinite order. Then every isomorphism $\mathcal{P}_{\mathrm{fin},1}(H) \to \mathcal{P}_{\mathrm{fin},1}(K)$ is cardinality-preserving.

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- Studying the quotients of $\{1, a\} \cdot \{1, b\}$, we obtain

$$g(ab) \in \{g(a)g(b), g(a)g(b)^{-1}, g(b)g(a)^{-1}\}.$$

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Proposition

If $a, b \in H$ have infinite order and $a^n = b^m$ for some $n, m \in \mathbb{Z} \setminus \{0\}$, then $g(ab) = g(a)g(b)$.

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Sketch of Proof

Set $x := g(a)$ and $y := g(b)$ and suppose by way of contradiction that $g(ab) \neq xy$. Then w.l.o.g. $g(ab) = xy^{-1}$ and

$$y^{n+m} = g(b^{n+m}) = g((ab)^n) = x^n y^{-n} = y^{m-n},$$

a contradiction.

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- We call an element $a \in H$ *reversed* if the restriction of f to the isomorphism $\mathcal{P}_{\text{fin},1}(\langle a \rangle) \to \mathcal{P}_{\text{fin},1}(\langle g(a) \rangle)$ is the reversion map.

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Proposition

If $H^\times \neq \{1\}$, then no element of H is reversed.

Non-reduced monoids

Theorem (R. 202?)

Let H, K be commutative, cancellative monoids with non-trivial unit groups. Then $\mathcal{P}_{\text{fin},1}(H) \simeq \mathcal{P}_{\text{fin},1}(K)$ if and only if $H \simeq K$.

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- In the reduced case, the situation is more complicated.
- H is called a *valuation monoid* if for every $x \in \mathsf{q}(H)$, we have $x \in H$ or $x^{-1} \in H$.

A counterexample

Theorem (R., 2026)

Let H and K be reduced valuation monoids with $\mathsf{q}(H) \simeq \mathsf{q}(K)$. Then $\mathcal{P}_{\mathrm{fin},1}(H) \simeq \mathcal{P}_{\mathrm{fin},1}(K)$.

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Let H and K be reduced valuation monoids with $\mathsf{q}(H) \simeq \mathsf{q}(K)$. Then $\mathcal{P}_{\mathrm{fin},1}(H) \simeq \mathcal{P}_{\mathrm{fin},1}(K)$.

- The proof relies on the following lemma

Lemma

Let H be a reduced valuation monoid and let $X \in \mathcal{P}_{\mathrm{fin},1}(G)$, where $G := \mathsf{q}(H)$. Then there is a unique $a \in G$ such that $aX = \{ax : x \in X\} \in \mathcal{P}_{\mathrm{fin},1}(H)$.

A counterexample

Theorem (R., 2026)

Let H and K be reduced valuation monoids with $\mathsf{q}(H) \simeq \mathsf{q}(K)$. Then $\mathcal{P}_{\mathrm{fin},1}(H) \simeq \mathcal{P}_{\mathrm{fin},1}(K)$.

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Example

Let $H := (\mathbb{Z} \times \mathbb{N}) \cup (\mathbb{N}_0 \times \{0\})$ and $K := \{(x, y) \in \mathbb{Z}^2 : y \leq \alpha x\}$, where $0 < \alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then H and K are non-isomorphic reduced valuation monoids with $\mathsf{q}(H) = \mathsf{q}(K) = (\mathbb{Z}^2, +)$.

The general case

- To fill in the gaps, we define

$$H_v := \{a \in H : \text{ for all } b \in H, ab^{-1} \in H \text{ or } a^{-1}b \in H\}$$

and ${H_v}^c := H \setminus H_v$.

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Theorem (R., 202?)

Let H, K be commutative and cancellative monoids. Then $\mathcal{P}_{\text{fin},1}(H) \simeq \mathcal{P}_{\text{fin},1}(K)$ if and only if $H \simeq K$ or H, K are reduced and K is isomorphic to ${H_v}^c \sqcup M$, where M is a valuation monoid with $\mathsf{q}(M) = \mathsf{q}(H_v)$.

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- What about cancellative non-commutative monoids?

Thank you for your attention!